INITIAL-BOUNDARY-VALUE PROBLEMS FOR MAGNETIC FLUX COMPRESSION*

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Summary

Elementary initial-boundary-value problems are solved for (i) a plane flux compression generator with field losses by diffusion, (ii) a cylindrical flux compressor with electromagnetic radiation, and (iii) electromagnetic pulse generation by means of magnetic flux pushers. Due to the moving boundary conditions, the analytical treatment of these problems leads to integral equations and infinite systems of coupled differential equations, respectively.

Introduction

Initial-boundary-value problems (IVBP) for the compression of the flux of typical magnetic field configurations by accelerated liners are treated by space- and time-dependent eigenfunctions, which take into account the temporal change of the field space due to the motion of the conductor interfaces. The derived solutions provide physical insight into the phenomena of flux compression and dilution, flux diffusion into accelerated conductors, and the occurance of electromagnetic (EM) radiation[1,2] during flux compression. Furthermore, the generation of EM wave pulses by means of accelerated flux pushers is discussed.

Plane Flux Compression and Dilution with Diffusion Losses

Figure 1 depicts an EM induction system with ideally conducting ($\sigma=\infty$) walls at $x=-b$ and $x=+d$, in which a slab of finite conductivity ($\sigma<\infty$) occupying the space $s(t) \leqslant x \leqslant s(t)+a$ is accelerated across a homogeneous, external magnetic field $\vec{B}_o=\{0,B_o,0\}$ to a velocity $\vec{v}=\{\dot{s}(t),0,0\}$, where $s(t=0)=s_o$, $-b<s_o<+d$. The motion $\dot{s}(t) \gtrless 0$ of the piston induces (i) EM fields $\vec{E}=\{0,0,E(x,t)\}$ and $\vec{B}=\{0,B(x,t),0\}$ in the conductor at $s(t) \leqslant x \leqslant s(t)+a$, and (ii) EM fields in the spaces behind $[E_1(x,t),B_1(t)]$ and in front $[E_2(x,t),B_2(t)]$ of it (Fig. 1). If the electric displacement current, $\partial\varepsilon\vec{E}/\partial t$ is neglected as usual,[3] the magnetic field $B(x,t)$ in the accelerated conductor is determined by the parabolic IBVP with moving boundaries at $x=s(t)$ and $x=s(t)+a$:

$$\partial B/\partial t+\dot{s}(t)\partial B/\partial x=\kappa\partial^2 B/\partial x^2 \quad , \quad s(t)<x<s(t)+a \quad , \tag{1}$$

$$B(x,t=0) = B_o \quad , \quad s_o<x<s_o+a \quad , \tag{2}$$

$$B(x=s(t),t) = B_1(t) \quad , \quad 0<t<\hat{t} \quad , \tag{3}$$

$$B(x=s(t)+a,t) = B_2(t) \quad , \quad 0<t<\hat{t} \quad , \tag{4}$$

where

$$d\{[s(t)+b]B_1(t)\}/dt = \kappa\partial B(x=s(t),t)/\partial x \quad , \tag{5}$$

$$B_1(t=0) = B_o \quad , \tag{6}$$

$$d\{[s(t)+a-d]B_2(t)\}/dt = \kappa\partial B(x=s(t)+a,t)/\partial x \quad , \tag{7}$$

$$B_2(t=0) = B_o \quad , \tag{8}$$

$\kappa=1/\mu\sigma$, and $\hat{t}$ is the flux compression time. Equations (5) and (7) result from the continuity $\vec{n}\times[\vec{E}]=\vec{0}$ of the tangential electric field at the rear (1) and front (2) interfaces of the conductor, where the conductor fields have been eliminated by means of Ohm's law, $\vec{E}=-\vec{v}\times\vec{B}+\kappa\nabla\times\vec{B}$, and the EM fields in the gaps (1,2) have been determined from $\nabla\times\vec{E}=-\partial\vec{B}/\partial t$ and $\nabla\times\vec{B}=\vec{0}$.

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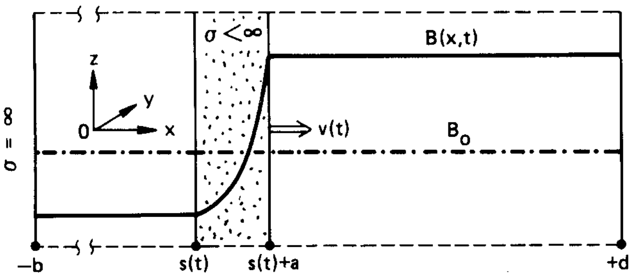

Fig. 1: EM induction model with accelerated conductor at $s(t) \leqslant x \leqslant s(t)+a$, and fixed liners at $x=-b$ and $x=+d$.

Expansion of $B(x,t)/B_o=\mathcal{B}(\xi,\tau)$ in space- and time-dependent eigenfunctions of

$$\xi=[x-s(t)/a] \ , \ \tau=\kappa t/a^2 \ ; \ 0\leqslant\xi\leqslant 1 \ , \ 0\leqslant\tau\leqslant\hat{\tau} \quad , \tag{9}$$

gives the solution of Eqs. (1) - (4) in dimensionless form

$$\mathcal{B}(\xi,\tau) = 1+2\pi\sum_{n=1}^{\infty} n\int_0^{\tau}\exp[-n^2\pi^2(\tau-\tau')]\{[\mathcal{B}_1(\tau')-1] - (-1)^n[\mathcal{B}_2(\tau')-1]\}d\tau'\cdot\sin n\pi\xi \quad . \tag{10}$$

Substitution of Eq. (10) into Eqs. (5) and (7) demonstrates that the gap fields $\mathcal{B}_1(\tau)=B_1(t)/B_o$ and $\mathcal{B}_2(\tau)=B_2(t)/B_o$ are determined by the coupled Volterra integro-differential equations $[\sigma(\tau)=s(t)/a,\ b=b/a,\ d=d/a]$

$$\frac{d}{d\tau}\{[\sigma(\tau)+b]\mathcal{B}_1(\tau)\} = \mathcal{B}_2(\tau)-\mathcal{B}_1(\tau) - \frac{d}{d\tau}\int_0^{\tau}K(\tau-\tau')[\mathcal{B}_1(\tau')-1]d\tau' + \frac{d}{d\tau}\int_0^{\tau}L(\tau-\tau')[\mathcal{B}_2(\tau')-1]d\tau' \quad , \tag{11}$$

$$\frac{d}{d\tau}\{[\sigma(\tau)+1-d]\mathcal{B}_2(\tau)\} = \mathcal{B}_2(\tau)-\mathcal{B}_1(\tau) + \frac{d}{d\tau}\int_0^{\tau}K(\tau-\tau')[\mathcal{B}_2(\tau')-1]d\tau' - \frac{d}{d\tau}\int_0^{\tau}L(\tau-\tau')[\mathcal{B}_1(\tau')-1]d\tau' \quad , \tag{12}$$

with

$$\mathcal{B}_1(\tau=0)=1 \quad , \quad \mathcal{B}_2(\tau=0)=1 \quad , \tag{13}$$

where

$$K(\tau-\tau') = 2\sum_{n=1}^{\infty}\exp[-n^2\pi^2(\tau-\tau')] \quad ,$$
$$L(\tau-\tau') = 2\sum_{n=1}^{\infty}(-1)^n\exp[-n^2\pi^2(\tau-\tau')] \quad , \tag{14}$$

are singular kernels of convolution type. In view of the extreme behavior of $K(\tau-\tau')$ and $L(\tau-\tau')$ for $\tau'=\tau$, the main contributions to the integrals arise for $\tau'\lesssim\tau$. For these reasons, Eqs. (11) - (13) have the approximate analytical solutions:

$$\mathcal{B}_1(\tau) = 1-\frac{\exp[-\int_0^{\tau}\omega(\tau')d\tau']\int_0^{\tau}\dot{\sigma}(\tau')\exp[+\int_0^{\tau'}\omega(\tau)d\tau]d\tau'}{\sigma(\tau)+b+p(\tau)-q(\tau)h_1(\tau)/h_2(\tau)} \quad , \tag{15}$$

$$\mathcal{B}_2(\tau) = 1+\frac{\exp[-\int_0^{\tau}\omega(\tau')d\tau']\int_0^{\tau}\dot{\sigma}(\tau')\exp[+\int_0^{\tau'}\omega(\tau)d\tau]d\tau'}{d-1-\sigma(\tau)+p(\tau)-q(\tau)h_2(\tau)/h_1(\tau)} \quad , \tag{16}$$

where

$$\omega(\tau) = \frac{b+d-1+2[p(\tau)+q(\tau)]}{h_1(\tau)h_2(\tau)-q(\tau)\{b+d-1+2[p(\tau)+q(\tau)]\}} \quad , \tag{17}$$

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$$h_1(\tau) = \sigma(\tau) + b + p(\tau) + q(\tau) \quad ,$$
$$h_2(\tau) = d - 1 - \sigma(\tau) + p(\tau) + q(\tau) \quad , \tag{18}$$

and

$$p(\tau) = +\int_0^\tau K(\tau - \tau')d\tau' = \frac{1}{3}[1 - \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-n^2\pi^2\tau)] \geqslant 0 \quad , \tag{19}$$

$$q(\tau) = -\int_0^\tau L(\tau - \tau')d\tau' = \frac{1}{6}[1 + \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\exp(-n^2\pi^2\tau)] \geqslant 0 \quad , \tag{20}$$

The results (10) and (15) - (16) are valid for diffusion times $t_D = \mu\sigma a^2 >> a/c$ (since the electric displacement current has been neglected)[1] and nonrelativistic, but otherwise arbitrary piston speeds $\dot{s}(t) = a\dot{\sigma}(\tau)$.

As distinguished from ideal flux compression theory, $\mathcal{B}_1(\tau)$ and $\mathcal{B}_2(\tau)$ remain finite for complete implosion $\tau \to \hat{\tau}$ of the left gap $[\sigma(\tau) \to -b]$ or right gap $[\sigma(\tau) \to d - 1]$, respectively (Fig. 1), since $p(\hat{\tau}) > 0$ and $q(\hat{\tau}) > 0$. Furthermore, $\mathcal{B}_1(\tau) \lesseqgtr 1$ and $\mathcal{B}_2(\tau) \gtreqless 1$ for $\dot{\sigma}(\tau) \gtreqless 0$ [compression (>) and dilution (<) of flux], and $\mathcal{B}_{1,2}(\tau) = 1$ for $\dot{\sigma}(\tau) = 0$. The current induced per unit width $\Delta y = 1\,\mathrm{m}$ in the moving conductor is

$$I(t) = (B_o/\mu)\mathcal{I}(\tau) \quad , \quad \mathcal{I}(\tau) = \mathcal{B}_2(\tau) - \mathcal{B}_1(\tau) \quad , \quad 0 \leqslant \tau \leqslant \hat{\tau} \quad . \tag{21}$$

As an illustration, consider a flux compression generator with $\sigma = 6{\times}10^7\,\mathrm{S/m}$ (Cu), $a = 10^{-2}\mathrm{m}$, $b = d \sim 10^{-1}\mathrm{m}$, $s_o = 0\,\mathrm{m}$, $\dot{s}(t) = v_o H(t)$, $v_o = 10^3\mathrm{m/sec}$, $\hat{t} = (d - a)/v_o \sim 10^{-4}$ sec. In this case, Eqs. (15) and (16) give $B_1(\hat{t}) \sim B_o/2$ and $B_2(\hat{t}) \sim 10^3 B_o$. The observed maximum field will be smaller due to nonlinear diffusion losses, which become important for intensities $B_2 \gtrsim B_{cr} = (\mu c_V T)^{1/2} = 6.5{\times}10^1\,\mathrm{T}$ for Cu at $T = 10^3\,{}^\circ\mathrm{K}$.

The order-of-magnitude of $B_1(t)$ and $B_2(t)$ does not change if the end planes $x = -b$ and $x = +d$ are formed by thick slabs of the same conductivity $\sigma < \infty$. For applications, it is noted that the $\sigma = \infty$ planes $x = -b$ and $x = +d$ can also be interpreted as symmetry planes for systems with four moving liners of conductivity σ.

Axial Flux Compression with EM Displacement

In flux compression experiments EM wave fields are observed,[4] which are due to the displacement current $\partial\epsilon\vec{E}/\partial t$. The implosion of a cylindrical liner (of conductivity $\sigma = \infty$ and initially at $R(t{=}0) = R_o$, Fig. 2) with a speed $\dot{R}(t) \leqslant 0$ against an initially homogeneous magnetic field $\vec{B}(r,t{=}0) = \{0,0,B_o\}$ excites by electromagnetic induction the wave fields

$$\vec{B}(r,t) = \frac{1}{r}\frac{\partial}{\partial r}[rA(r,t)]\vec{e}_z \quad , \quad \vec{E}(r,t) = -\frac{\partial A(r,t)}{\partial t}\vec{e}_\theta \quad . \tag{22}$$

The azimuthal vector potential $A(r,t)$ is determined by the hyperbolic IBVP with moving boundary at $r = R(t)$:

$$\frac{\partial^2 A}{\partial t^2} = c^2[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial A}{\partial r}) - \frac{A}{r^2}] \; , \quad 0 < r < R(t) \quad , \tag{23}$$

$$A(r,t{=}0) = \frac{1}{2}B_o r \quad , \quad 0 < r < R_o \quad , \tag{24}$$

$$\partial A(r,t{=}0)/\partial t = 0 \quad , \quad 0 < r < R_o \quad , \tag{25}$$

$$A(r{=}0,t) = 0 \quad , \quad 0 < t < \hat{t} \quad , \tag{26}$$

$$A(r{=}R(t),t) = \frac{1}{2}B_o R_o^2/R(t) \; , \quad 0 < t < \hat{t} \quad , \tag{27}$$

where $R_o = -\int_0^{\hat{t}}\dot{R}(t)dt$ defines the hypothetical time $\hat{t}$ of complete implosion, $R(t{=}\hat{t}) = 0$. Equations (24), (26) and (27) specify that $B(r,t{=}0) = B_o$, $E(r,t{=}0) = 0$, and $\int_0^{R(t)} B(r,t)2\pi r dr = \pi R_o^2 B_o$.

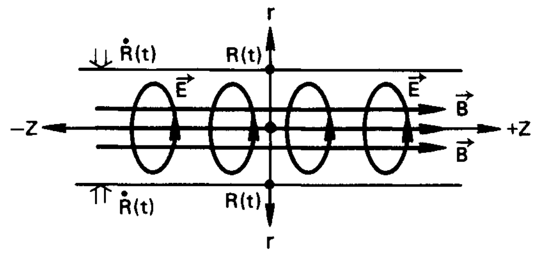

Fig. 2: EM fields $\vec{E}(r,t)$ and $\vec{B}(r,t)$ induced by imploding liner at $r = R(t)$.

Equation (26) holds since the flux within the circle $r = 0$ is necessarily zero. The hyperbolic wave fields $\psi(r,t)$ are separated from the quasi-static elliptic compression fields by means of the ansatz

$$A(r,t) = \psi(r,t) + \frac{1}{2}B_o[R_o/R(t)]^2 r \quad , \quad 0 \leqslant r \leqslant R(t) \quad . \tag{28}$$

Equation (28) transforms the IBVP (23) - (27) into an IBVP with a source field $Q(r,t) = B_o[\ddot{R}(t)/R(t) - 3\dot{R}(t)^2/R(t)^2][R_o/R(t)]^2 r$ and homogeneous boundary conditions, $\psi(r{=}0,t) = 0$ and $\psi(r{=}R(t),t) = 0$. For the latter reason, $\psi(r,t)$ can be expanded in the r- and t-dependent eigenfunctions $f_k(r,t)$,

$$\psi(r,t) = 2[R(t)]^{-2}\sum_{k=1}^{\infty}\psi_k(t)[J_o(\alpha_k)]^{-2}f_k(r,t) \quad ,$$
$$\psi_k(t) = \int_0^{R(t)}\psi(r,t)f_k(r,t)r dr \quad , \tag{29}$$

where

$$f_k(r,t) = J_1(\alpha_k r/R(t)) \quad , \quad J_1(\alpha_k) = 0, \; k = 1,2,3,\cdots\infty \quad . \tag{30}$$

The eigenvalues α_k follow from the boundary condition $\psi(r{=}R(t),t) = 0$, whereas $\psi(r{=}0,t) = 0$, since $J_1(0) = 0$. By means of this extended Fourier method, the solution for $\psi(r,t)$ is found which gives, under consideration of Eqs. (22) and (28), the EM field solutions in the imploding cylindrical cavity $0 \leqslant r \leqslant R(t)$:

$$A(r,t) = \frac{1}{2}B_o[\frac{R_o}{R(t)}]^2 r + \frac{2}{R(t)^2}\sum_{k=1}^{\infty}\frac{\psi_k(t)}{J_o(\alpha_k)^2}J_1(\frac{\alpha_k r}{R(t)}) \quad , \tag{31}$$

and

$$B(r,t) = B_o[\frac{R_o}{R(t)}]^2 + \frac{2}{R(t)^3}\sum_{k=1}^{\infty}\frac{\alpha_k\psi_k(t)}{J_o(\alpha_k)^2}J_o(\frac{\alpha_k r}{R(t)}) \quad , \tag{32}$$

$$E(r,t) = \dot{R}(t)B_o[\frac{R_o}{R(t)}]^2\frac{r}{R(t)} + \frac{2\dot{R}(t)}{R(t)^3}\sum_{k=1}^{\infty}\frac{\psi_k(t)}{J_o(\alpha_k)^2}\frac{\alpha_k r}{R(t)}J_o(\frac{\alpha_k r}{R(t)})$$
$$+ \frac{2}{R(t)^3}\sum_{k=1}^{\infty}\frac{[\dot{R}(t)\psi_k(t) - R(t)\dot{\psi}_k(t)]}{J_o(\alpha_k)^2}J_1(\frac{\alpha_k r}{R(t)}) \quad . \tag{33}$$

The Fourier amplitudes $\psi_k(t)$ of the hyperbolic wave fields are determined by the infinite set of coupled differential equations [derived from the inhomogeneous wave equation for $\psi(r,t)$],

$$\frac{d^2\psi_k}{dt^2} - 2\frac{\dot{R}(t)}{R(t)}\frac{d\psi_k}{dt} + [\frac{(\alpha_k c)^2}{R(t)^2} - 2\frac{\ddot{R}(t)}{R(t)} + (2 - \frac{1}{3}\alpha_k^2)\frac{\dot{R}(t)^2}{R(t)^2}]\psi_k$$
$$= Q_k(t) - \sum_{j\neq k}^{\infty}[M_{kj}(t)\psi_j + N_{kj}(t)\frac{d\psi_j}{dt}] \quad , \tag{34}$$

with

$$\psi_k(t{=}0) = 0 \quad , \quad d\psi_k(t{=}0)/dt = 0 \quad , \quad k = 1,2,3,\cdots\infty \quad , \tag{35}$$

where

$$Q_k(t) = B_o R_o^2\{3\dot{R}(t)^2/R(t) - \ddot{R}(t)\}\, J_o(\alpha_k)/\alpha_k \quad , \quad (36)$$

and

$$M_{kj}(t) = 2\frac{\dot{R}(t)^2}{R(t)^2}\left[\frac{\ddot{R}(t)\,R(t)}{\dot{R}(t)\,\dot{R}(t)} - 5 - \frac{4\alpha_j^2}{\alpha_k^2-\alpha_j^2}\right]\frac{\alpha_j\alpha_k}{\alpha_k^2-\alpha_j^2}\left[\frac{J_o(\alpha_k)}{J_o(\alpha_j)} - \delta_{kj}\right] \quad , \quad (37)$$

$$N_{kj}(t) = 4\frac{\dot{R}(t)}{R(t)}\frac{\alpha_j\alpha_k}{\alpha_k^2-\alpha_j^2}\left[\frac{J_o(\alpha_k)}{J_o(\alpha_j)} - \delta_{kj}\right] \quad . \quad (38)$$

The coupling of the wave modes k in Eq. (34) is due to the boundary motion $\dot{R}(t)$, since $M_{kj}(t) \neq 0$ and $N_{kj}(t) \neq 0$ for $\dot{R}(t) \neq 0$.

Equations (31) - (33) indicate that the complete hyperbolic solutions are the sum of (i) quasi-static (S) fields and (ii) transient wave (R) fields (Fourier series). By Eq. (34), the "radiation" fields are relativistically small since

$$\psi_k(t) \to 0 \quad , \quad \dot{R}(t)^2/c^2 \to 0 \quad . \quad (39)$$

Accordingly, flux compression theory without displacement current is approximately valid for nonrelativistic liner speeds, $\dot{R}(t)^2 \ll c^2$. Since $\psi_k(t) \sim B_o(\dot{R}(t)/c)^2 R_o^2 \times R(t)$ by Eq. (34) and $\vec{B}^2/2\mu = (\vec{B}_S + \vec{B}_R)^2/2\mu$, the radiative energy density is of the order $\vec{B}_S \cdot \vec{B}_R/\mu \sim (B_o^2/\mu)(\dot{R}(t)/c)^2$. For these reasons, an explanation of magnetic flux compression during gravitational collapse of magnetic stars with speeds $|\dot{R}(t)| \lesssim c$ must be based on Maxwell's equations with displacement current.[4]

EM Wave Pulses from Flux Pushers

Theoretical investigations indicate that EM waves are excited at the surface of conductors which are accelerated in magnetic fields.[1,2] In cylindrical flux compression experiments, the emission of EM pulses from the openings of the cylinder ends has been observed.[4] These explosion generated EM pulses have a low characteristic frequency (Fourier spectrum) $\omega \sim \Delta t^{-1}$ since the duration of the explosive conductor motion is of the order $\Delta t \sim 10^{-4}$sec.

As an illustrative model for the inductive generation of EM pulses, consider a plane copper piston of thickness d with its front surface initially at $x = 0$, which borders on a homogeneous magnetic field $\vec{B}_o = \{0, B_o, 0\}$ filling the half space $x > 0$ (Fig. 3). At time $t = 0$, this piston is pushed with a velocity $\vec{v} = \{\dot{a}(t), 0, 0\}$ into the B_o-field region $x \geq a(t) > 0$. The duration Δt of the impulsive piston motion is assumed to be small compared with the EM diffusion time $t_D = \mu_o \sigma d^2 \sim 10^{-2}$sec (for $\sigma = 6\times10^7$S/m and $d = 10^{-2}$m), so that the piston can be treated as a perfect conductor ($\sigma = \infty$). For a piston of area $\Delta A = 1\,m^2$, the EM pressure force F, its work ΔW along a piston path $\Delta x = 1$ m, and the generated power P for a shock wave driven piston with speed $\bar{v} \sim 10^4$m/sec are

$$F = \Delta A B_o^2/\mu \cong 10^6\text{N}, \quad \Delta W = F\Delta x \cong 10^6\text{J}, \quad P \cong 10^{10}\text{J/sec}$$

if $B_o = (4\pi/10)^{1/2} \cong 1.1$ T. The duration of the piston motion is $\Delta t \sim \Delta x/\bar{v} \sim 10^{-4}$sec. The expanded work ΔW must appear in the form of an EM energy pulse.

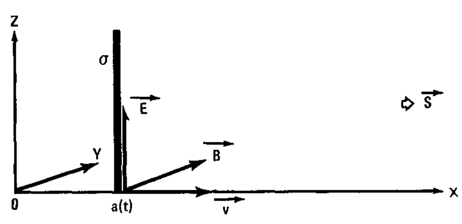

Fig. 3: Plane flux pusher at $x = a(t)$ with transverse magnetic field $\vec{B}$ and EM pulse $\vec{S}$.

For the plane magnetic flux pusher in Fig. 3, the EM fields are of the form, $\vec{B} = \{0, -\partial A(x,t)/\partial x, 0\}$ and $\vec{E} = \{0, 0, -\partial A(x,t)/\partial t\}$. The vector potential is determined by the IBVP with a moving boundary condition at the piston front face $x = a(t)$:

$$\partial^2 A/\partial t^2 = c^2 \partial^2 A/\partial x^2 \quad , \quad a(t) < x < \infty \quad , \quad (40)$$

$$A(x, t{=}0) = -B_o x \quad , \quad 0 < x < \infty \quad , \quad (41)$$

$$\partial A(x, t{=}0)/\partial t = 0 \quad , \quad 0 < x < \infty \quad , \quad (42)$$

$$[\partial A/x,t)/\partial t + \dot{a}(t)\partial A(x,t)/\partial x]_{x=a(t)} = 0 \quad , \quad t > 0 \quad , \quad (43)$$

Equations (41) - (43) consider that $B(x,t{=}0) = B_o$, $E(x,t{=}0) = 0$, and $\vec{E} + \vec{v}\times\vec{B} = \vec{0}$ at the piston ($\sigma = \infty$) surface $x = a(t)$. The solution of Eqs. (40) - (43) is by the Laplace transform method

$$A(x,t) = -B_o x + \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)e^{s(t-x/c)}ds \quad (44)$$

where the amplitude function F(s) satisfies by Eq. (43) the integral equation

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} sF(s)e^{s[t-a(t)/c]}ds = \dot{a}(t)[1 - \dot{a}(t)/c]^{-1}B_o \quad . \quad (45)$$

Equation (45) can be solved analytically for certain piston motions a(t), e.g., for the Heaviside step velocity model,

$$\dot{a}(t) = v_o H(t) \quad , \quad F(s) = v_o B_o(1 - v_o/c)^{-1}s^{-2} \quad . \quad (46)$$

From Eqs. (44) and (46) one obtains the solutions for the EM fields and the Poynting vector $\vec{S} = \vec{E}\times\vec{B}/\mu$:

$$A(x,t) = -B_o x + v_o B_o(1 - \frac{v_o}{c})^{-1}(t - \frac{x}{c})H(t - \frac{x}{c}) \quad , \quad x > v_o t \quad , \quad (47)$$

$$E(x,t) = -v_o B_o(1 - \frac{v_o}{c})^{-1}H(t - \frac{x}{c}) \quad , \quad x > v_o t \quad , \quad (48)$$

$$B(x,t) = B_o + \frac{v_o}{c}(1 - \frac{v_o}{c})^{-1}B_o H(t - \frac{x}{c}) \quad , \quad x > v_o t \quad ,$$

$$= 0 \quad , \quad x < v_o t \quad , \quad (49)$$

and

$$S = -EB/\mu = S_o + \tilde{S} \quad (50)$$

where

$$S_o(x,t) = v_o\frac{B_o^2}{\mu}(1 - \frac{v_o}{c})^{-1}H(t - \frac{x}{c}) \quad , \quad x > v_o t \quad ,$$

$$= 0 \quad , \quad x < v_o t \quad , \quad (51)$$

$$\tilde{S}(x,t) = \frac{v_o}{c}v_o\frac{B_o^2}{\mu}(1 - \frac{v_o}{c})^{-2}H(t - \frac{x}{c}) \quad , \quad x > v_o t \quad ,$$

$$= 0 \quad , \quad x < v_o t \quad . \quad (52)$$

It is seen that the energy flow density S consists of a "mixed" (static-transient) term $S_o = -EB_o/\mu$ and a true wave term $\tilde{S} = -E\tilde{B}/\mu$, where $\tilde{S} \sim (v_o/c)S_o$ if $v_o \ll c$. The hypothetical limits, $S_o \to \infty$ and $\tilde{S} \to \infty$ for $v_o \to c$, are remarkable. For the above B_o, $v_o = \bar{v}$, and Δt values, Eqs. (51) and (52) give ($\Delta A = 1\,m^2$)

$$S_o\Delta A\Delta t \cong 10^6\text{J}, \text{ and } \tilde{S}\Delta A\Delta t = (1/3)\times10^2\text{J}$$

since $1 - v_o/c \cong 1$. Accordingly, the main energy flow is in S_o, but the true wave contribution $\tilde{S}$ represents also a significant, measurable effect. The efficiency of the production of the wave pulse $\tilde{S}$ is of the order $v_o/c \sim 10^{-4}/3$, which is comparable to the efficiency $\sim 10^{-4}$ of the radiation production in pulsed x-ray tubes.[5]

By the same method, the EM wave pulses generated by means of more complicated flux pushers (Figs. 4,5) can be evaluated.[6]

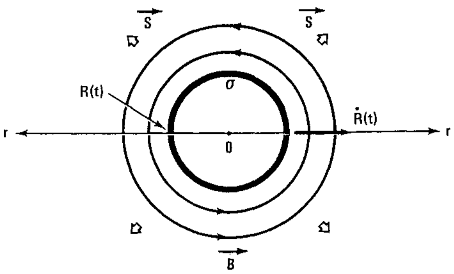

Fig. 4: Cylindrical flux pusher at $r = R(t)$ with azimuthal magnetic field $\vec{B}$ and EM pulse $\vec{S}$.

The radial EM pulse generator in Fig. 4 consists of a cylindrical liner ($\sigma \to \infty$) of initial (external) radius $R(t{=}0) = R_0$, which is exploded to a radius $R(t) > R_0$ at time $t > 0$. The liner carries a current $I(t) = 2\pi R(t) \times j_z^*(t)$ on its surface $r = R(t)$ with $I(t{=}0) \sim 10^6$A. The pulsed liner motion $R(t)$ transverse to the magnetic field $B_\theta(r,t)$ of $I(t)$ excites an EM energy pulse in radial direction,

$$S_r = -E_z(r,t)B_\theta(r,t)/\mu > 0 \;\; , \;\; R(t) < r < R_0 + ct \quad , \qquad (53)$$

since the induced electric field is $E_z(r,t) \sim -\dot{R}(t) \times B_\theta(r,t) < 0$. The liner functions for a limited pulse time Δt (depending on its initial thickness ΔR_0), since the conductor becomes too thin and too transparent at a critical expansion radius $R_{cr} = R(t{=}\Delta t)$.

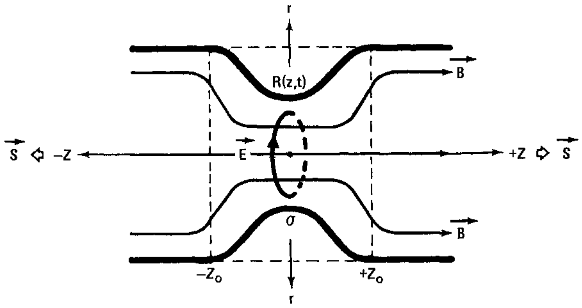

Fig. 5: Cylindrical flux pusher at $r = R(z,t)$ with locally deformed magnetic field $\vec{B}$ and EM pulses $\vec{S}$.

The axial EM pulse generator in Fig. 5 consists of a cylindrical liner ($\sigma \to \infty$) of initial (inner) radius $R(z,t{=}0) = R_0$ and contains an initially homogeneous magnetic field $B_z(r,z,t{=}0) = B_0$ in the axial direction, where $B_0 \sim 1$ T. This liner is imploded with the speed $\partial R(z,t)/\partial t < 0$ within the axial region $-z_0 \leq z \leq +z_0$. The resulting compression of the axial magnetic flux induces an electric wave pulse

$$E_\theta(r,z,t) = -\frac{1}{2\pi r}\int_0^r \frac{\partial B_z(r,z,t)}{\partial t}\, 2\pi r dr < 0 \;\; , \;\; 0 < |z| < ct \;\; , \qquad (54)$$

since $\partial B_z(r,z,t)/\partial t > 0$. The energy flux of the induced EM fields is in the axial directions,

$$S_z = -E_\theta(r,z,t)B_r(r,z,t)/\mu \gtrless 0 \text{ for } z \gtrless 0 \;,\; 0 < |z| < ct \;\; , \qquad (55)$$

since $B_r(r,z,t) \gtrless 0$ for $z \gtrless 0$. Thus, EM wave pulses are ejected from both ends of the cylinder during magnetic flux compression[4] (Fig. 5).

The generation of EM wave fluxes $\vec{S}$ by means of flux pushers would be very efficient if relativistic liner motions $v \lesssim c$ were realizable. For this reason, the transformation of chemical energy of explosives, via work against the EM back pressure of an initially (quasi-) static magnetic field, into EM radiation represents an interesting, but not readily solvable problem. If Ohmic losses are negligible ($\Delta t/t_D \to 0$), this energy transformation mechanism follows directly from Poynting's theorem, according to which the work $\partial W/\partial t$ of the flux pusher per unit time equals the EM energy flow $\vec{S}$ through the surface ($d\vec{a}$) of the system ($\sigma \to \infty$),

$$\partial W/\partial t = -\mu^{-1}\oint \vec{E}\times\vec{B}\cdot d\vec{a}.$$

Appendix: Derivation of Eq. (34)

Substitution of Eq. (28) into Eq. (23) results in the inhomogeneous wave equation for $\psi(r,t)$,

$$\frac{\partial^2\psi}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \frac{\psi}{r^2}\right] + Q(r,t) \;\; , \;\; 0 < r < R(t) \;\; , \qquad (A1)$$

where

$$Q(r,t) = B_0\left[\frac{\ddot{R}(t)}{R(t)} - 3\frac{\dot{R}(t)^2}{R(t)^2}\right]\left[\frac{R_0}{R(t)}\right]^2 r \qquad (A2)$$

is a known source field. The Fourier amplitudes $\psi_k(t)$ are determined by multiplication of Eq. (A1) with the eigenfunctions (30) and subsequent integration over the area $\pi R(t)^2$,

$$\int_0^{R(t)} \frac{\partial^2\psi}{\partial t^2} f_k r dr = c^2\int_0^{R(t)} \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \frac{\psi}{r^2}\right] f_k r dr + \int_0^{R(t)} Q f_k r dr \;. \qquad (A3)$$

Partial integrations, under consideration of the boundary values $\psi(r,t) = 0$ and $f_k(r,t) = 0$ for $r = 0,\ R(t)$, yield

$$\int_0^{R(t)} \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \frac{\psi}{r^2}\right] f_k r dr = -[\alpha_k/R(t)]^2\psi_k(t) \quad , \qquad (A4)$$

and

$$\int_0^{R(t)} \frac{\partial^2\psi}{\partial t^2} f_k r dr = \frac{d^2\psi_k(t)}{dt^2} - \int_0^{R(t)} \left[2\frac{\partial\psi}{\partial t}\frac{\partial f_k}{\partial t} + \psi\frac{\partial^2 f_k}{\partial t^2}\right] r dr \;\; , \qquad (A5)$$

Equation (A3) gives, under consideration of Eqs. (A4) and (A5), the differential equation for $\psi_k(t)$:

$$\frac{d^2\psi_k(t)}{dt^2} + [\alpha_k c/R(t)]^2\psi_k(t) - Q_k(t) = \Omega_k(t) \qquad (A6)$$

where

$$\Omega_k(t) = 2\sum_{j=1}^{\infty} J_0(\alpha_j)^{-2}\int_0^{R(t)} \left[2\frac{\partial}{\partial t}\left(\frac{\psi_j f_j}{R(t)^2}\right)\frac{\partial f_k}{\partial t} + \left(\frac{\psi_j f_j}{R(t)^2}\right)\frac{\partial^2 f_k}{\partial t^2}\right] r dr \;. \qquad (A7)$$

The Bessel integrals in Eq. (A7) can be evaluated in closed form. Thus, one finds Eq. (34) from Eq. (A6).

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